Math 3235 Probability Theory

3/16/23

$X \quad Y$

$u(X,Y) = U \qquad v(X,Y) = V$

$f_{X,Y}(x,y)$ joint p.d.f. of X and Y. Find the joint p.d.f. of U and V. (Chapter 6.5 of the Textbook).

$X = x(U,V)$

$Y = y(U,V)$ Inverse function

$$\begin{tikzcd} (X,Y) \arrow[rr, bend left, "{u,v}"] & & (U,V) \arrow[ll, bend left, "{x,y}"] \end{tikzcd}$$

For example:

$u = x+y \qquad v = \dfrac{x}{x+y}$

in Term of random variables

$U = X+Y \qquad V = \dfrac{X}{X+Y}$

$\Downarrow$

$X = UV \qquad Y = U(1-V)$

$x = uv \qquad y = u(1-v)$

$$J(u,v) = \begin{pmatrix} \dfrac{\partial x(u,v)}{\partial u} & \dfrac{\partial x(u,v)}{\partial v} \\[2ex] \dfrac{\partial y(u,v)}{\partial u} & \dfrac{\partial y(u,v)}{\partial v} \end{pmatrix}$$

jacobian matrix

$J(u,v)$ is a 2×2 matrix whose entries are functions of u, v.

$$x = uv \qquad y = u(1-v)$$

$$\frac{\partial x}{\partial u} = v \qquad \frac{\partial x}{\partial v} = u$$

$$\frac{\partial y}{\partial u} = (1-v) \qquad \frac{\partial y}{\partial v} = -u$$

$$j(u,v) = \begin{pmatrix} v & u \\ 1-v & -u \end{pmatrix}$$

$$J(u,v) = \left| \det j(u,v) \right|$$

Reminder:

if $\quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$\det A = a_{11} a_{22} - a_{21} a_{12}$$

$$J(u,v) = \left| \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right|$$

In our example we get

$$J(u,v) = \left| -uv - u(1-v) \right| = |u|$$

Theorem:

X, Y are cont. r.v. with joint p.d.f. $f_{X,Y}(x,y)$.

T is a mapping from $\mathbb{R}^2 \to \mathbb{R}^2$

$T(x,y) = (u(x,y), v(x,y))$ and

T is bijective. Then I can write $T^{-1}(u,v) = (u(x,y), v(x,y))$

and

$$U = u(X,Y) \qquad V = v(X,Y)$$

$$f_{U,V}(u,v) = f_{X,Y}(x(u,v), y(u,v))\, J(u,v)$$

Assume that X and Y have joint p.d.f. given by

$$f_{X,Y}(x,y) = \begin{cases} e^{-x-y} & x, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$U = X + Y \qquad V = \frac{X}{X+Y}$$

Observe that

$$f_X(x) = \int_0^\infty e^{-x-y}\,dy = e^{-x}$$

$$f_Y(y) = \int_0^\infty e^{-x-y}\,dx = e^{-y}$$

X and Y are indep. exp with par 1.

$$e^{-x-y} = f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$$U = X+Y \qquad V = \frac{X}{X+Y}$$

$$J(u,v) = u$$

$$f_{X,Y}(x(u,v), y(u,v)) = e^{-u}$$

$$f_{U,V}(u,v) = u e^{-u}$$

$u = x+y$ since both x and y are positive Than $x+y \geq 0$.

$$0 \leq \frac{x}{x+y} = v \leq 1$$

Finally the joint p.d.f of U and V is

$$f_{U,V}(u,v) = \begin{cases} u e^{-u} & u>0,\ 0<v<1 \\ 0 & \text{otherwise} \end{cases}$$

Marginals:

$$f_U(u) = \int_0^1 u e^{-u}\, dv = u e^{-u} \qquad u>0$$

$$f_V(v) = \int_0^{\infty} u e^{-u}\, du = 1 \qquad 0 \leq v \leq 1$$

U is Gamma $(2, 1)$ while V is uniform in $[0,1]$.

Moreover U and V are independent.

Idea of the proof of the Theorem.

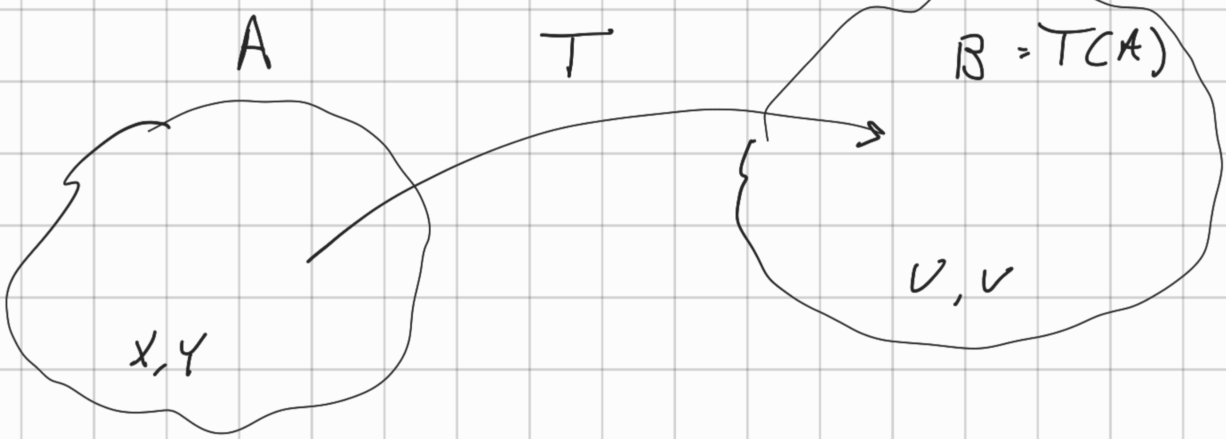
$B \in \mathbb{R}^2$ and we want to know

$$\mathbb{P}((U,V)\in B) = \iint_B f_{U,V}(u,v)\,du\,dv$$

If $B = T(A)$ that is $A = T^{-1}(B)$

it follows that

$$\mathbb{P}((U,V)\in B) = \mathbb{P}((X,Y)\in A)$$



$$\mathbb{P}((X,Y)\in A) = \iint_A f_{X,Y}(x,y)\,dx\,dy =$$

(change of variable $u = u(x,y)$, $v = v(x,y)$)

$$= \iint_{B = T^{-1}(A)} f_{X,Y}(x(u,v), y(u,v))\, J(u,v)\, du\, dv$$

For every subset $B \subset \mathbb{R}^2$ we have

$$\iint_B f_{U,V}(u,v)\, du\, dv =$$

$$\iint_B f_{X,Y}(x(u,v), y(u,v))\, J(u,v)\, du\, dv$$

$$\Downarrow$$

$$f_{U,V}(u,v) = f_{X,Y}(x(u,v), y(u,v))\, J(u,v)$$

Example

$X \quad Y$

$$f_{X,Y}(x,y) = \begin{cases} e^{-y} & y > x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) = \int_x^{\infty} e^{-y}\,dy = -e^{-y}\Big|_x^{\infty} = e^{-x}$$

$$f_Y(y) = \int_0^{y} e^{-y}\,dx = y e^{-y}$$

$$U = X \qquad V = Y - X$$

$$X = U \qquad Y = U + V$$

$$f_{U,V}(u,v) = e^{-y(u,v)} J(u,v) =$$

$$= e^{-(u+v)} J(u,v)$$

$$J(u,v) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$J(u,v) = 1$$

$$f_{U,V}(u,v) = \begin{cases} e^{-u-v} & u>0 \quad v>0 \\ 0 & \text{otherwise} \end{cases}$$

Expected Values of j. dist. r.v.

If X, Y are continuous r.v. and $g: \mathbb{R}^2 \to \mathbb{R}$, we can consider

$$Z = g(X, Y)$$

Z is a new continuous r.v.

Theorem:

$$\mathbb{E}(Z) = \mathbb{E}(g(X, Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\, dx\, dy$$

$$g(X, Y) = X$$

$$\mathbb{E}(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X,Y}(x,y)\, dx\, dy =$$

$$= \int_{-\infty}^{\infty} x \left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \right) dx =$$

$$= \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$\mathbb{E}(Y) = \int_{-\infty}^{\infty} y \, f_Y(y) \, dy$$

$$\mathbb{E}(aX + bY) =$$

$$\iint (ax + by) \, f_{X,Y}(x,y) \, dx \, dy =$$

$$a \iint x \, f_{X,Y}(x,y) \, dx \, dy \; +$$
$$b \iint y \, f_{X,Y}(x,y) \, dx \, dy =$$

$$a \, \mathbb{E}(X) + b \, \mathbb{E}(Y)$$

What about the product if X and Y are independ.

$$\mathbb{E}(XY) = \iint xy \, f_{XY}(x,y) \, dx \, dy =$$

$$= \iint xy \, f_X(x) \, f_Y(y) \, dx \, dy =$$

$$= \int_{-\infty}^{\infty} y\, f_Y(y)\, dy \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$= \mathbb{E}(X)\,\mathbb{E}(Y).$$

If $X \perp\!\!\!\perp Y \Rightarrow \mathbb{E}(XY) = \mathbb{E}(X)\,\mathbb{E}(Y)$

$$\not\Leftarrow$$

We define

$$\operatorname{cov}(X,Y) = E(XY) - \mathbb{E}(X)\,\mathbb{E}(Y)$$

$$\operatorname{corr}(X,Y) = \rho_{X,Y} = \frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{var}(X)\operatorname{var}(Y)}}$$

$$\operatorname{cov}(aX+b,\ cY+d) =$$

$$ac \operatorname{cov}(X,Y)$$

$$\mathbb{E}(XY) = \mathbb{E}(X)\,\mathbb{E}(Y) \not\Rightarrow X \perp\!\!\!\perp Y$$

Theorem: if for every functions $g: \mathbb{R} \to \mathbb{R}$ and $h: \mathbb{R} \to \mathbb{R}$

$$\mathbb{E}(g(X)\,h(Y)) = \mathbb{E}(g(X))\,\mathbb{E}(h(Y))$$

Then

X and Y are independent.

Proof:

$$g(x) = \begin{cases} 1 & x < a \\ 0 & \text{otherwise} \end{cases}$$

$$h(y) = \begin{cases} 1 & y < b \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}(g(X)\,h(Y)) =$$

$$\int_{-\infty}^{a}\int_{-\infty}^{b} f_{X,Y}(x,y)\,dx\,dy =$$

$$\mathbb{P}(X \leq a \;\&\; Y \leq b)$$

$$\mathbb{E}(g(X)) = \mathbb{P}(X \leq a)$$

$$\mathbb{E}(h(Y)) = \mathbb{P}(Y \leq b)$$

$$\Downarrow$$

$$\mathbb{P}(X \leq a \ \& \ Y \leq b) =$$

$$\mathbb{P}(X \leq a)\,\mathbb{P}(Y \leq b)$$

$$F(a, b) = F(a)\,F(b).$$